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**AN ABS ALGORITHM FOR A CLASS OF SYSTEMS OF
STOCHASTIC LINEAR EQUATIONS**

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AN ABS ALGORITHM FOR A CLASS OF SYSTEMS OF STOCHASTIC LINEAR EQUATIONS

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Abstract. This paper is to explore a model of the ABS Algorithms for dealing with a class of systems of linear stochastic equations $A\xi = \eta$ satisfying $\eta \sim N_m(v, I_m)$. It is shown that the iteration step α_i is $N(V, \pi)$ and approximation solutions is $\xi_i \sim N_n(U, \Sigma)$ for this algorithm model. And some properties of (V, π) and (U, Σ) are given.

Key words: ABS algorithm, stochastic linearly system of equations, distribution, probability.

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1 Introduction

It is well known, that in stochastic programming, stochastic linear equations being of the form

$$A(w)\xi(w) = \eta(w) \quad (1.1)$$

or

$$A\xi(w) = \eta(w) \quad (1.2)$$

with $\xi(w)$ $\eta(w)$ are random vectors. If the constraint equation of the mathematical programming, $Ax = b$ contain the random variables, then the stochastic linear equation has the form (1.1) or (1.2).

In this paper we try explore an application of the classical ABS algorithms to solve system indicated by (1.2), in other words, we attempt to establish an model of algorithm, called ABS-S defined in Section 3 for solving (1.2). To this end, it is necessary to recall the ABS algorithms.

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Consider the general linear systems, where $\text{rank}(A)$ is arbitrary,

$$Ax = b \quad (1.3)$$

or

$$a_i^T x = b_i, \quad i = 1, \dots, m \quad (1.4)$$

where $x \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, $m \leq n$ and $A \in \mathbb{R}^{m \times n}$, $A = (a_1, a_2, \dots, a_m)^T$.

The class of ABS algorithms originally for solving (1.3) or (1.4) was introduced by [AbBs84] and [AbSp89]. The iterate scheme of the basic ABS class of algorithm is defined as follows:

Basic ABS Class of Algorithms: see, [AbBs84], [AbBs89]:

(A) Initialization.

Give an arbitrary vector $x_1 \in \mathbb{R}^n$, and an arbitrarily nonsingular matrix $H_1 \in \mathbb{R}^{n,n}$.

Set $i = 1$ and iflag=0.

(B) Computer two quantities.

Compute

$$\begin{aligned} s_i &= H_i a_i \\ \tau_i &= \tau^T e_i = a_i^T x_i - b^T e_i \end{aligned}$$

(C) Check the compitability of the system of linear equations.

If $s_i \neq 0$ then goto (D).

If $s_i = 0$ and $\tau_i = 0$ then set

$$\begin{aligned} x_{i+1} &= x_i \\ H_{i+1} &= H_i \end{aligned}$$

and goto (F), the i -th equation is a linear combination of the previous equations.

Otherwise stop, the system has no solution.

(D) Computer the search vector $p_i \in \mathbb{R}^n$ by

$$p_i = H_i^T z_i \quad (1.5)$$

where z_i , the parameter of Broyden, is arbitrary save that

$$z_i^T H_i a_i \neq 0 \quad (1.6)$$

(E) Update the approximation of the solution x_i by

$$x_{i+1} = x_i - \alpha_i p_i \quad (1.7)$$

where the stepsize α_i is computed by

$$\alpha_i = \tau_i / a_i^T p_i \quad (1.8)$$

If $i = m$ stop; x_{m+1} solves the system.

(F) Update the (Abaffian) matrix H_i . Compute

$$H_{i+1} = H_i - H_i a_i w_i^T H_i / w_i^T H_i a_i \quad (1.9)$$

where $w_i \in \mathbb{R}^n$, the parameter of Abaffy, is arbitrary save for the condition

$$w_i^T H_i a_i = 1 \text{ or } \neq 0 \quad (1.10)$$

(G) Increment the index i by one and goto (B).

We define n by i matrices A_i , W_i and P_i by

$$A_i = (a_1, \dots, a_i)^T, \quad W_i = (w_1, \dots, w_i), \quad P_i = (p_1, \dots, p_i) \quad (1.11)$$

Some properties of the above recursion, see for instance, Abaffy and Spedicato (1989), [AbSp 89], are listed below that are the basic formulae for use later on.

a. Implicit factorization property

$$A_i^T P_i = L_i \quad (1.12)$$

with L_i nonsingular lower triangular.

b. Null space characterizations

$$\begin{aligned} \mathcal{N}(H_{i+1}) &= \mathcal{R}(A_i^T), \quad \mathcal{N}(H_{i+1}^T) = \mathcal{R}(W_i), \\ \mathcal{N}(A_i) &= \mathcal{R}(H_{i+1}^T) \end{aligned} \quad (1.13)$$

where \mathcal{N} = Null and \mathcal{R} =Range.

- c. The linear variety containing all solutions to $Ax = b$ consists of the vectors of the form

$$x = x_{t+1} + H_{t+1}^T q \quad (1.14)$$

where $q \in \mathbb{R}^n$ is arbitrary.

This paper are organized as follows. Section 2 gives some basic definitions , operations that will be used below. We establish ABS algorithm of stochastic linear equations in Section 3. In section 4, we prove that ξ_i, α_i have some good properties,such as ξ_i, α_i , are both normal distributions, and give iterative formula of their expectation and variance. In section 5, a example is given to show the ABS algorithm of stochastic linear equations. In section 6 α_i is discussed and same results are also given.

2 Preliminaries

Consider a class of systems of stochastic linear equations being of the form

$$A\xi(w) = \eta(w) \quad (2.1)$$

In this paper, we give the ABS algorithm under the condition that $\eta(w)$ is a m -dimensional normal distribution. And it is shown that the iteration step length α_i is $N(V, \pi)$ and the iteration solution ξ_i is $N_n(U, \Sigma)$. the (V, π) and (U, Σ) are determined by four iterative formulae. Finally, it is proven that ξ_{i+1} is a solution of the first i equations and the step length α_i is discussed.

Given a random experiment with a sample space Ω , a function ξ that assigns to each element ω in Ω one and only one real number $\xi(\omega) = x$ is called a random variable. The space of ξ is the set of real number $\{x : x = \xi(\omega) \in \Omega\}$, where $\omega \in \Omega$ means the element ω belongs to the set Ω .

If $\xi_1, \xi_2, \dots, \xi_n$ are the n random variable over $(\Omega, \mathcal{A}, \mathcal{P})$, then the vector function $(\xi_1, \xi_2, \dots, \xi_n)$ are Ω is called a n -dimensional random vector are $(\Omega, \mathcal{A}, \mathcal{P})$.

If $\xi_i (i = 1, 2, \dots, n)$ are $N(0, 1)$ and independent random variables, then

$$\eta = (\eta_1, \eta_2, \dots, \eta_m)^T = A_{mn}\xi + \mu_{m1}$$

is called m -dimensional random vectors, whose probability density function or distribution function is simply called m -dimensional normal distribution, denote $\eta \sim$

$N_m(\mu, AA^T)$. It is easy to know that $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T \sim N_n(0, I_n)$.

Definition 2.1 *It is said that the random variable ξ is equal to the random variable η , denoted by $\xi = \eta$. if distributions of ξ and η are the same and $E\xi = E\eta$ and $D\xi = D\eta$.*

Definition 2.2 *Consider the system of random linear equations $A\xi = \eta$, where η is given and ξ is to be found. ξ is said to be a solution of this system of ξ satisfies this system in the sense of Definition1.*

If

$$\xi = \eta + a, a \in \mathcal{R}$$

then

$$E\xi = E\eta + a, D\xi = D\eta.$$

3 ABS-S Algorithm for solving System of Stochastic Linear Equations

Consider a system of the stochastic linear equations

$$A\xi = \eta \tag{3.1}$$

where $\eta = (\eta_1, \eta_2, \dots, \eta_m)^T$ is m -dimensional stochastic vector, $A = (a_1, a_2, \dots, a_m)^T \in R^{m,n}$

the ABS-S algorithm is defined, based on the basic ABS algorithm as follows. **ABS-S**

Algorithm: for solving systems of stochastic linear equations

(A1) Initialization.

Give an arbitrary stochastic vector $\xi_1 \in \mathbb{R}^n$, and an arbitrarily nonsingular matrix $H_1 \in R^{n,n}$.

Set $i = 1$ and iflag=0.

(B1) Computer two quantities.

Compute

$$\begin{aligned}s_i &= H_i a_i \\ \tau_i &= a_i^T \xi_i - \eta_i\end{aligned}$$

(C1) Check the compatibility of the system of linear equations.

If $s_i \neq 0$ then goto (D).

If $s_i = 0$ and $P(\tau_i = 0) = 1$ then set

$$\begin{aligned}\xi_{i+1} &= \xi_i \\ H_{i+1} &= H_i\end{aligned}$$

and go to (F), the i -th equation is a linear combination of the previous equations.

Otherwise stop, the system has no solution.

(D1) Compute the search vector $p_i \in \mathbb{R}^n$ by

$$p_i = H_i^T z_i \tag{3.2}$$

where z_i , the parameter of Broyden, is arbitrary save that

$$z_i^T H_i a_i \neq 0 \tag{3.3}$$

(E1) Update the random approximation ξ_i of a solution to (3.3) by

$$\xi_{i+1} = \xi_i - \alpha_i p_i \tag{3.4}$$

where the stepsize α_i is computed by

$$\alpha_i = \frac{\tau_i}{a_i^T p_i} \tag{3.5}$$

If $i = m$ stop; ξ_{m+1} solves the system(2.1).

(F1) Update the (Abaffian) matrix H_i . Compute

$$H_{i+1} = H_i - \frac{H_i a_i^T w_i H_i}{w_i^T H_i a_i} \tag{3.6}$$

where $w_i \in \mathbb{R}^n$, the parameter of Abaffy, is arbitrary save for the condition

$$w_i^T H_i a_i = 1 \text{ or } \neq 0 \tag{3.7}$$

(G1) Increment the index i by one and go to (B1).

4 The properties of α_i and ξ_i

Let $\eta \sim N_m(v, I_m)$, i.e, $\eta_i \sim N(v_i, 1), i = 1, 2, \dots, m$ it is shown that the step length α_i and the iterative solution ξ_i obtained in the ABS-S algorithm given above. Are both normal distributions, that is to say.

Theorem 4.1

$$\begin{aligned}\tau_1 &= a_1^T \xi_1 - \eta_1 \sim N(a_1^T \xi_1 - v_1, 1) \\ \alpha_1 &= \frac{\tau_1}{a_1^T p_1} \sim N\left(\frac{a_1^T \xi_1 - v_1}{a_1^T p_1}, \frac{1}{(a_1^T p_1)^2}\right)\end{aligned}$$

Proof. From $\eta \sim N_m(V, I_m)$, we know that $\eta_i \sim N(v_i, 1), i = 1, 2, \dots, m$. according to the linearity of the normal distribution. again, we see that τ_1 is normal distribution and $\eta \sim N(v, I_m)$, $\eta_i \sim N(v_i, 1), i = 1, 2, \dots, m$

$$\begin{aligned}E\tau_1 &= E(a_1^T \xi_1 - \eta_1) \\ &= E(a_1^T \xi_1) - E(\eta_1) \\ &= a_1^T \xi_1 - v_1 \\ D\tau_1 &= D(a_1^T \xi_1 - \eta_1) \\ &= D(a_1^T \xi_1) + D(\eta_1) \\ &= 1.\end{aligned}$$

$$\begin{aligned}E\alpha_1 &= E\frac{\tau_1}{a_1^T p_1} \\ &= \frac{E\tau_1}{a_1^T p_1} \\ &= \frac{a_1^T \xi_1 - v_1}{a_1^T p_1}. \\ D\alpha_1 &= D\left(\frac{\tau_1}{a_1^T p_1}\right) \\ &= \frac{D\tau_1}{a_1^T p_1} \\ &= \frac{1}{(a_1^T p_1)^2}.\end{aligned}$$

Therefore

$$\begin{aligned}\alpha_1 &= \frac{\tau_1}{a_1^T p_1} \sim N\left(\frac{a_1^T \xi_1 - v_1}{a_1^T p_1}, \frac{1}{(a_1^T p_1)^2}\right) \\ \tau_1 &= a_1^T \xi_1 - \eta_1 \sim N(a_1^T \xi_1 - v_1, 1)\end{aligned}$$

The proof is completed.

Theorem 4.2

$$\xi_2 \sim N_n\left(\xi_1 - \frac{a_1^T \xi_1 - v_1}{a_1^T p_1} p_1, \frac{p_1 p_1^T}{(a_1^T p_1)^2}\right)$$

Proof. By using Theorem 4.1, we have

$$\alpha_1 \sim N\left(\frac{a_1^T \xi_1 - v_1}{a_1^T p_1}, \frac{1}{(a_1^T p_1)^2}\right)$$

According to the linearity of the normal distribution. Again, we see that $\xi_2 = \xi_1 - \alpha_1 p_1$ is normal distribution, and

$$\begin{aligned}E\xi_2 &= E(\xi_1 - \alpha_1 p_1) = E(\xi_1) - E(\alpha_1 p_1) \\ &= \xi_1 - E(\alpha_1) p_1 = \xi_1 - \frac{a_1^T \xi_1 - v_1}{(a_1^T p_1)} p_1 \\ D\xi_2 &= D(\xi_1 - \alpha_1 p_1) \\ &= D(\xi_1) + D(\alpha_1 p_1) \\ &= p_1 (D\alpha_1) p_1^T \\ &= \frac{p_1 p_1^T}{(a_1^T p_1)^2}\end{aligned}$$

Therefore

$$\xi_2 \sim N_n\left(\xi_1 - \frac{a_1^T \xi_1 - v_1}{a_1^T p_1} p_1, \frac{p_1 p_1^T}{(a_1^T p_1)^2}\right)$$

The proof is completed.

Theorem 4.3 $a_i p_j = 0, \quad i < j$

Proof. see [AbBs 84], [AbSp 89]

Theorem 4.4

$$\tau_i \sim N(a_i^T \xi_1 - a_i^T \sum_{j=1}^{i-1} (E\alpha_j) p_j - v_i, 1 + a_i^T \sum_{j,k=1}^{i-1} \text{cov}(\alpha_j p_j, \alpha_k p_k) a_i), i \geq 2$$

Proof. The ABS-S algorithm gives rise to

$$\tau_i = a_i^T \xi_i - \eta_i$$

$$\xi_{i+1} = \xi_i - \alpha_i p_i$$

therefore

$$\begin{aligned} \tau_i &= a_i^T \xi_i - \eta_i \\ &= a_i^T (\xi_{i-1} - \alpha_{i-1} p_{i-1}) - \eta_i \\ &= \dots \\ &= a_i^T (\xi_1 - \alpha_1 p_1 - \alpha_2 p_2 - \dots - \alpha_{i-1} p_{i-1}) - \eta_i \\ &= a_i^T (\xi_1 - \sum_{j=1}^{i-1} (\alpha_j p_j)) - \eta_i \end{aligned}$$

From the iterative process, we know that α_i is normal distribution, $\eta_i \sim N(v_i, 1)$

hence τ_i is normal distribution, and

$$\begin{aligned} E\tau_i &= E(a_i^T (\xi_1 - \sum_{j=1}^{i-1} (\alpha_j p_j)) - \eta_i) \\ &= E(a_i^T (\xi_1 - \sum_{j=1}^{i-1} (\alpha_j p_j))) - E\eta_i \\ &= a_i^T (\xi_1 - \sum_{j=1}^{i-1} (E\alpha_j) p_j) - v_i \\ D\tau_i &= D(a_i^T (\xi_1 - \sum_{j=1}^{i-1} (\alpha_j p_j)) - \eta_i) \\ &= D(a_i^T (\xi_1 - \sum_{j=1}^{i-1} (\alpha_j p_j))) + D\eta_i - 2cov(a_i^T (\xi_1 - \sum_{j=1}^{i-1} (\alpha_j p_j)), \eta_i) \\ &= a_i^T D(\xi_1 - \sum_{j=1}^{i-1} (\alpha_j p_j)) a_i + 1 + 2cov(a_i^T \sum_{j=1}^{i-1} (\alpha_j p_j), \eta_i) \\ &= a_i^T [\sum_{j=1}^{i-1} D(\alpha_j p_j)] a_i + \sum_{j \neq k} cov(\alpha_j p_j, \alpha_k p_k) a_i + 1 \\ &= a_i^T [\sum_{j,k=1}^{i-1} cov(\alpha_j p_j, \alpha_k p_k)] a_i + 1. \end{aligned}$$

The proof is completed.

Theorem 4.5

$$\alpha_i \sim N\left(\frac{a_i^T \xi_1 - a_i^T \sum_{j=1}^{i-1} (E\alpha_j) p_j - v_i}{a_i^T p_i}, \frac{a_i^T [\sum_{j,k=1}^{i-1} cov(\alpha_j p_j, \alpha_k p_k)] a_i + 1}{(a_i^T p_i)^2}\right), i \geq 2$$

Proof. According to $\alpha_i = \frac{\tau_i}{a_i^T p_i}$ and the linearity of the normal distribution, it is easy to known that the property holds.

The proof of theorem is completed.

Theorem 4.6

$$\xi_{i+1} \sim N_n(\xi_1 - \sum_{j=1}^{j=i} (E\alpha_j)p_j, \sum_{j,k=1}^i \text{cov}(\alpha_j p_j, \alpha_k p_k))$$

or

$$\xi_{i+1} \sim N_n(E\xi_i - (E\alpha_i)p_i, D\xi_i + p_i D(\alpha_i)p_i^T - 2\text{cov}(\xi_i, \alpha_i p_i)), i \geq 2$$

Proof.

$$\begin{aligned} \xi_i &= \xi_{i-1} - \alpha_{i-1}p_{i-1} \\ &= \xi_{i-2} - \alpha_{i-2}p_{i-2} - \alpha_{i-1}p_{i-1} \\ &= \dots \\ &= \xi_1 - \alpha_1p_1 - \alpha_2p_2 - \dots - \alpha_{i-1}p_{i-1} \\ &= \xi_1 - \sum_{j=1}^{j=i-1} \alpha_j p_j. \end{aligned}$$

By $\alpha_j (j = 1, 2, \dots, i-1)$ is normal distribution, we have ξ_i is normal distribution, and

$$\begin{aligned} E\xi_i &= E(\xi_1 - \sum_{j=1}^{j=i-1} (\alpha_j)p_j) \\ &= E\xi_1 - \sum_{j=1}^{j=i-1} E(\alpha_j)p_j \\ &= \xi_1 - \sum_{j=1}^{j=i-1} E(\alpha_j)p_j \\ D\xi_i &= D(\xi_1 - \sum_{j=1}^{j=i-1} (\alpha_j)p_j) \\ &= D\xi_1 - \sum_{j=1}^{j=i-1} D(\alpha_j)p_j \\ &= \sum_{j=1}^{j=i-1} D(\alpha_j p_j) + \sum_{j \neq k} \text{cov}(\alpha_j p_j, \alpha_k p_k) \\ &= \sum_{j=1}^{j=i-1} D(\alpha_j p_j) + \sum_{j \neq k} \text{cov}(\alpha_j p_j, \alpha_k p_k) \\ &= \sum_{j,k=1}^{i-1} \text{cov}(\alpha_j p_j, \alpha_k p_k). \end{aligned}$$

The proof is completed.

Theorem 4.7 ξ_{i+1} is the solution of the fist i equations

Proof. Because of

$$\xi_{i+1} \sim N_n(\xi_1 - \sum_{j=1}^{j=i} E(\alpha_j)p_j, \sum_{j,k=1}^i \text{cov}(\alpha_j p_j, \alpha_k p_k))$$

When $l < i$,

$$\begin{aligned}
a_i^T \xi_{i+1} &= a_i^T (\xi_i - \alpha_i p_i) \\
&= a_i^T \xi_i - a_i^T \frac{\tau_i}{a_i^T p_i} p_i \\
&= a_i^T \xi - \tau_i \\
&= \eta_i. \\
a_l^T \xi_{i+1} &= a_l^T (\xi_i - \alpha_i p_i) \\
&= a_l^T \xi_i - a_l^T \alpha_i p_i \\
&= a_l^T \xi_i.
\end{aligned}$$

Hence

$$\begin{aligned}
E(a_l^T \xi_{i+1}) &= E(a_l^T \xi_i), D(a_l^T \xi_{i+1}) = D(a_l^T \xi_i) \\
a_l^T \xi_{i+1} &\sim N(v_l, 1), l \leq i.
\end{aligned}$$

The proof is completed.

Theorem 4.8 *The solution of $A\xi = \eta$ is obtained in the finite steps by using ABS algorithm.*

Proof. According to the definition (2.1) of the solution of the system of the random linear equations $A\xi = \eta$, it is easy to see that the theorem holds.

The proof of Theorem is completed.

5 Example

Stochastic linear equation

$$\begin{pmatrix} 1 & 3 & -1 & 0 & 2 & 0 \\ 0 & -2 & 4 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} \xi^{(1)} \\ \xi^{(2)} \\ \xi^{(3)} \\ \xi^{(4)} \\ \xi^{(5)} \\ \xi^{(6)} \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} \quad (5.1)$$

$\eta_1 \sim N(6, 1)$, $\eta_2 \sim N(12, 1)$, $\eta_3 \sim N(2, 1)$ are independent each other. Our aim is determining the distribution of $\xi_i = (\xi_i^{(1)}, \xi_i^{(2)}, \xi_i^{(3)}, \xi_i^{(4)}, \xi_i^{(5)}, \xi_i^{(6)})$ to such that ξ_i is solution of equation, then we obtain the distribution of ξ_i using ABS-S algorithm:

Let

$$\xi_1 = (1, 1, 1, 1, 1, 1)^T, H_1 = I_6, z_i = a_i, \omega_i = a_i, i = 1, 2, 3$$

Then

step 1 $i=1$

$$\tau_1 = a_1^T \xi_1 - \eta_1 = 5 - \eta_1 \sim N(-1, 1)$$

$$\alpha_1 = \frac{5 - \eta_1}{15} \sim N(-\frac{1}{15}, \frac{1}{15^2})$$

$$p_1 = H_1^T z_1, \omega_1 = a_1$$

$$\xi_2 = \xi_1 - \alpha_1 p_1 = (\frac{10 + \eta_1}{15}, \frac{\eta_1}{5}, \frac{20 - \eta_1}{15}, 1, \frac{5 + 2\eta_1}{15}, 1)^T \sim N_6(U_1, \sigma_1)$$

where

$$U_1 = (\frac{16}{15}, \frac{18}{15}, \frac{16}{15}, 1, \frac{17}{15}, 1)^T$$

$$\sigma_1 = \frac{1}{15^2} \begin{pmatrix} 1 & 3 & -1 & 0 & 2 & 0 \\ 3 & 9 & -3 & 0 & 6 & 0 \\ -1 & -3 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 6 & -2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$H_2 = H_1 - \frac{H_1 a_1^T \omega_1 H_1}{\omega_1^T H_1 a_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

step 2 i=2

$$\tau_2 = a_2 \xi_2 - \eta_2 = \frac{90 - 10\eta_1 - 15\eta_2}{15} \sim N\left(\frac{1}{3}, \frac{13}{9}\right)$$

$$p_2 = H_2^T z_2 = (10, -2, 4, 1, 0, 0)^T$$

$$\alpha_2 = \frac{\tau_2}{a_2^T p_2} = \frac{90 - 10\eta_1 - 15\eta_2}{315} \sim N\left(\frac{1}{63}, \frac{13}{3969}\right)$$

$$\xi_3 = \xi_2 - \alpha_2 p_2$$

$$= \left(\frac{-740 - 121\eta_1 + 150\eta_2}{315}, \frac{190 + 43\eta_1 - 30\eta_2}{315}, \frac{40 + 19\eta_1 + 60\eta_2}{315}, \frac{220 + 10\eta_1 + 15\eta_2}{315}, \frac{5 + 2\eta_1}{15}, 1 \right)^T$$

$$H_3 = H_2 - \frac{H_2 a_2^T \omega_2 H_2}{\omega_2^T H_2 a_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & 2 & 1 & 0 & 0 & 0 \\ -1.5 & 0.5 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

step 3 i=3

$$\tau_3 = a_3 \xi_3 - \eta_3 = \frac{-615 - 237\eta_1 + 180\eta_2 - 315\eta_3}{315} \sim N(-1.61, 1.89)$$

$$\alpha_3 = \frac{615 + 237\eta_1 - 180\eta_2 + 315\eta_3}{630}$$

$$p_3 = H_3^T z_3 = (-1, 2, 1, 0, -2, 1)^T$$

$$\xi_4 = \xi_3 - \alpha_3 p_3$$

$$= \left(\frac{-865 + 479\eta_1 + 120\eta_2 + 315\eta_3}{630}, \frac{-850 - 388\eta_1 + 300\eta_2 - 630\eta_3}{630}, \frac{-535 - 199\eta_1 + 300\eta_2 - 315\eta_3}{630}, \right.$$

$$\frac{440 + 20\eta_1 + 30\eta_2}{630}, \frac{1440 + 558\eta_1 - 360\eta_2 + 630\eta_3}{630}, \frac{15 - 237\eta_1 + 180\eta_2 - 315\eta_3}{630})^T$$

$$\xi_4 \sim N(U, \Sigma)$$

where

$$U = (6.47, -1.33, 1.97, 1.46, 2.74, 0.195)^T$$

$$\Sigma = \begin{pmatrix} 0.525 & -1.011 & -0.331 & 0.037 & 0.958 & -0.475 \\ -1.011 & 1.75 & 0.886 & 0.007 & 1.695 & 0.865 \\ -0.331 & 0.886 & 0.538 & 0.008 & 1.015 & 0.503 \\ 0.037 & 0.007 & 0.008 & 0.003 & -0.004 & 0.002 \\ 0.958 & 1.695 & 1.015 & -0.004 & 1.998 & -0.99 \\ -0.495 & 0.865 & 0.503 & 0.002 & -0.99 & 0.472 \end{pmatrix}$$

It is clear that the following formulae hold

$$a_3^T \xi_4 = \eta_3, \quad a_2^T \xi_4 = \eta_2, \quad a_1^T \xi_4 = \eta_1$$

Then ξ_4 is a solution of stochastic linear equation (5.1).

6 Conclusion and Discussion

6.1 On the step length

Since

$$\alpha_i \sim N\left(\frac{a_i^T \xi_1 - v_i}{a_i^T p_i}, \frac{1 + a_i^T \sum_{j,k=1}^{i-1} \text{cov}(\alpha_j p_j, \alpha_k p_k) a_i}{(a_i^T p_i)^2}\right), \quad i \geq 2$$

Let

$$E\alpha_i = \frac{a_i^T \xi_1 - v_i}{a_i^T p_i}$$

$$D\alpha_i = \frac{1 + a_i^T \sum_{j,k=1}^{i-1} \text{cov}(\alpha_j p_j, \alpha_k p_k) a_i}{(a_i^T p_i)^2}$$

Then

$$\alpha_i \sim N(E\alpha_i, D\alpha_i)$$

therefore

$$P(E\alpha_i - D\alpha_i \leq \alpha_i \leq E\alpha_i + D\alpha_i) = 0.6827 \quad (6.1)$$

$$P(E\alpha_i - 2D\alpha_i \leq \alpha_i \leq E\alpha_i + 2D\alpha_i) = 0.9545 \quad (6.2)$$

$$P(E\alpha_i - 3D\alpha_i \leq \alpha_i \leq E\alpha_i + 3D\alpha_i) = 0.9973 \quad (6.3)$$

Therefore, we know that

$$\alpha_i \in [E\alpha_i - 3D\alpha_i, E\alpha_i + 3D\alpha_i]$$

that is to say, α_i are determined by the initial vector ξ_1 , and the first component v_1 of η as well as $a_i^T p_i$. In the practical applications, one can chooses arbitrary are formula of (6.1) to (6.3) according to the need and the given creditable degree.

6.2 main results

1. system of stochastic linear equations (1.2) with $\eta \sim N_m(v, I_m)$ can be solved by ABS-S algorithm with the random solution ξ under assumption.
2. The solution ξ_i generated by ABS-S algorithm is normal distribution.
3. The step length α_i generated by ABS-S algorithm is normal distribution.
4. Since the matrix A is non-random, H_i, p_i generated by ABS-S algorithm are non-random, thus the ABS-S algorithm to solve $A\xi = \eta$ have the same properties as that of the basic ABS algorithm to solve $Ax = b$

6.3 On problems are to be further studied

1. The first problem are that when η is an other distributions, whether $A\xi = \eta$ have solution and if so, whether it can be resolved by ABS-S algorithm.
2. The second problem is that when a is random matrix, whether $A\xi = \eta$ have solution and if so, whether it can be resolved by ABS-S algorithm.
3. The third problem is that when η is a function of random variable, problem are that whether $A\xi = \eta$ have solution and if so, whether it can be resolved by ABS-S algorithm.

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